B. Sc. part - II (H), paper - III 17-7-2020.

Tangent and Normals. problems.

Q.1. Find the condition that the line $x\cos\alpha + y\sin\alpha = p$ may touch the curve $\frac{x^m}{a^m} + \frac{y^m}{b^m} = 1$.

Soln The given eqn of the curve is $\frac{x^m}{a^m} + \frac{y^m}{b^m} = 1$ —— (1)

Differentiating with respect to x, we get

$$m\frac{x^{m-1}}{a^m} + m\frac{y^{m-1}}{b^m}\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -\frac{b^m}{a^m}\cdot\frac{x^{m-1}}{y^{m-1}}$$

At the point (x_1, y_1) of the curve, $\frac{dy}{dx} = -\frac{b^m}{a^m}\frac{x_1^{m-1}}{y_1^{m-1}}$

The eqn of the tangent to the curve (1) at the point (x_1, y_1) is

$$y - y_1 = \left(\frac{dy}{dx}\right)_{(x_1,y_1)}(x - x_1) - \frac{b^m}{a^m}\frac{x_1^{m-1}}{y_1^{m-1}}(x - x_1)$$

$$\Rightarrow \frac{y_1^{m-1}}{b^m}(y - y_1) + \frac{x_1^{m-1}}{a^m}(x - x_1) = 0$$

$$\Rightarrow \frac{y y_1^{m-1}}{b^m} - \frac{y_1^m}{b^m} + \frac{x x_1^{m-1}}{a^m} - \frac{x_1^m}{a^m} = 0$$

$$\Rightarrow \frac{x x_1^{m-1}}{a^m} + \frac{y y_1^{m-1}}{b^m} = \frac{x_1^m}{a^m} + \frac{y_1^m}{b^m}$$

Since, the point (x_1, y_1) lies on the curve (1) and therefore

$$\frac{x_1^m}{a^m} + \frac{y_1^m}{b^m} = 1 \quad \text{—— (2)}$$

$$\therefore \frac{x x_1^{m-1}}{a^m} + \frac{y y_1^{m-1}}{b^m} = 1 \quad \text{—— (3)}$$

By question the eqn of the tangent is

$$\frac{x}{p}\cos\alpha + \frac{y}{p}\sin\alpha = 1 \quad \text{—— (4)}$$

Comparing (3) and (4), we get

$$\frac{x_1^{m-1}}{a^m} = \frac{1}{p}\cos\alpha \text{ and } \frac{y_1^{m-1}}{b^m} = \frac{1}{p}\sin\alpha$$

$$\therefore x_1 = \frac{a^{\frac{m}{m-1}}(\cos\alpha)^{\frac{1}{m-1}}}{p^{\frac{1}{m-1}}} \text{ and } y_1 = \frac{b^{\frac{m}{m-1}}(\sin\alpha)^{\frac{1}{m-1}}}{p^{\frac{1}{m-1}}}$$

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Substituting in (3), we get

$$\frac{a^{\frac{m^2}{m-1}}(\cos\alpha)^{\frac{m}{m-1}}}{a^m p^{\frac{m}{m-1}}} + \frac{b^{\frac{m^2}{m-1}}(\sin\alpha)^{\frac{m}{m-1}}}{b^m p^{\frac{m}{m-1}}} = 1$$

$$\Rightarrow (a\cos\alpha)^{\frac{m}{m-1}} + (b\sin\alpha)^{\frac{m}{m-1}} = p^{\frac{m}{m-1}}$$

Which is the required condition.

Q.2. Prove that the condition that $x\cos\alpha + y\sin\alpha = p$ should $x^m y^n = a^{m+n}$ is $p^{m+n} m^m n^n = (m+n)^{m+n} a^{m+n} \sin^n\alpha \cdot \cos^m\alpha$.

Soln: The eqn of the tangent at (x,y) is

$$Y - y = \frac{dy}{dx}(X - x)$$

Taking the logarithmic differentiation of $x^m y^n = a^{m+n}$ w.r.to x,

$$m\frac{1}{x} + n\frac{1}{y}\cdot\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -\frac{m}{n}\frac{y}{x}$$

$\therefore$ ~~Eqn of the tangent~~ at (x,y) is

$$Y - y = -\frac{m}{n}\cdot\frac{y}{x}(X - x)$$

$$\Rightarrow myX + nxY = mxy + nxy \quad \text{——— (1)}$$

Now, if $X\cos\alpha + Y\sin\alpha = p$ ——— (2) be the tangent to the curve at (x,y) then (1) and (2) must be identical.

Comparing, $\frac{my}{\cos\alpha} = \frac{nx}{\sin\alpha} = \frac{(m+n)xy}{p}$

$\therefore \frac{mp}{(m+n)\cos\alpha} = x$ and $\frac{np}{(m+n)\sin\alpha} = y$

But $x^m y^n = a^{m+n}$

$$\therefore \frac{m^m p^m}{(m+n)^m \cos^m\alpha} \times \frac{n^n p^n}{(m+n)^n \sin^n\alpha} = a^{m+n}$$

$$\Rightarrow p^{m+n} m^m n^n = (m+n)^{m+n} a^{m+n} \sin^n\alpha \cos^m\alpha$$

Proved